Fourth Quarter Project

Honors Precalculus

Instructions: This is our final chance to delve in, explore, ponder, sit back, reflect, and marvel- and write up your results in a final project.

I think you will enjoy this last investigation, as it connects trigonometry, functions, and infinite series.

The **rubric** for evaluation is on the back of this page. Your work will count as your final unit test grade.

You will have five days of class time to complete this project. You may use homework time if you wish. Your work must be done in Google Docs.

A note on collaboration: You **may** discuss the content of this project with Mr. O'Brien, other students or anyone else but be sure to acknowledge any assistance received. You should be careful not to copy any results from anyone else (especially no copy-paste!). Your final write-up must be your own.

Final Draft Due: Midnight Tuesday, June 16th (Please email me if it is ready to be marked earlier.)

Name:________________________________

Project Rubric

Category	Poor	Fair	Good	Excellent
Presentation (10%) • Is the paper neat? • Is the paper typed? • Is the paper done in an orderly manner? • Are the diagrams easy to read?				
Mathematical precision and completeness (60%) • Are the solutions complete? • Are the solutions correct? • Has there been a correct use of mathematical notation?				
Verbal explanations (30%) • Are the explanations correct? • Are the explanations complete and precise? • Is there correct use of grammar and spelling?				

General Comments:

Final Grade:________

The Point Les Trip

One of the most intriguing concepts in mathematics is the idea of infinity. Infinite things sometimes run counter to what we expect - that's one reason they are fun to consider. One amazing aspect of infinite series is that sometimes they have a finite sum. This means it is possible to add water to a jar forever, using progressively smaller amounts, and yet never have the jar overflow – what an intriguing concept! Another version of this same concept is the idea of taking a trip involving an infinite number of segments that has a total length which is finite. This is the idea we will explore in this project. In this project, we will compare the length of the trip taken by point Ed, consisting of four segments, and the length of the trip taken by point Les, consisting of an infinite number of segments.

1. In rectangle ABCD, $AB = 2$, and $BC = 1$. (See Figure 1.) The path that point Ed will take consists of traveling around the exterior of the rectangle from point A to B to C to D and then back to A. What is the distance of this trip?

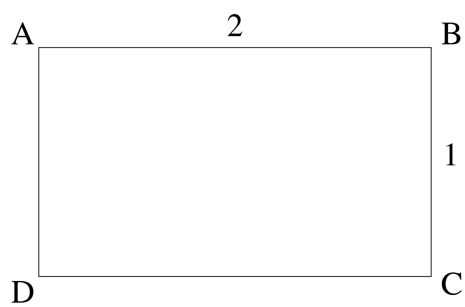

Figure 1

2. "The Road Les Traveled" is more complicated. Point Les travels along the bold path shown in Figure 2. He goes from A to B to E to F to G to $\cdots$ to C to A. Note that a series of similar right triangles are formed. The goal is to find the total distance traveled by point Les.

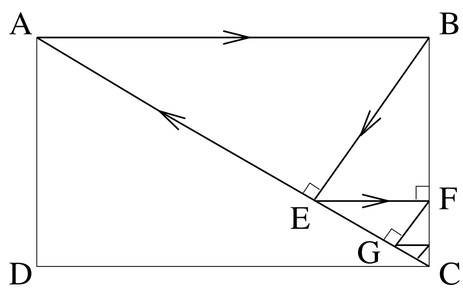

Figure 2

(a) The first segment of point Les's trip is $\overline{AB}$ which has length 2. The second segment of point Les's trip is $\overline{BE}$. Note that $\overline{BE}$ is perpendicular to diagonal $\overline{AC}$, which has length $\sqrt{5}$. (See Figure 3.) Explain why $\triangle BCE$ is similar to $\triangle ACB$. Use this to find the length of $\overline{BE}$.

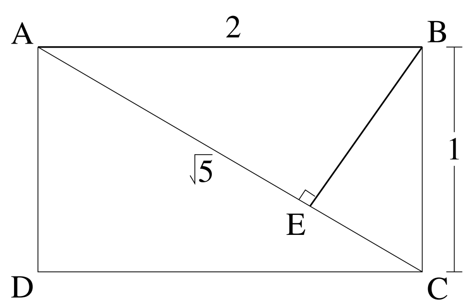

Figure 3

(b) From point E, Les will travel along line segment $\overline{EF}$ which is perpendicular to $\overline{BC}$. (See Figure 4.) Explain why $\triangle ECF \sim \triangle BCE$. Use this to find the length of $\overline{EF}$.

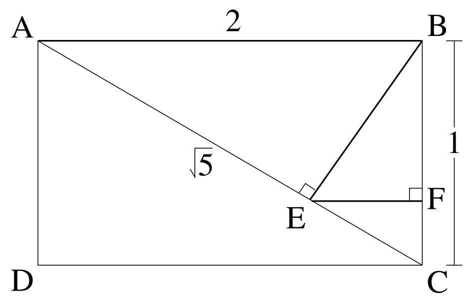

Figure 4

(c) From point F, we will travel along line segment $\overline{FG}$ which is perpendicular to diagonal $\overline{AC}$. (See Figure 5.) As before, $\triangle EFG \sim \triangle ECF$. Find the length of $\overline{FG}$. This process will continue indefinitely. While the number of segments for this trip is infinite, we will soon show that the total length of this trip is finite. After reaching point C, Les will travel along the diagonal, $\overline{AC}$, back to point A.

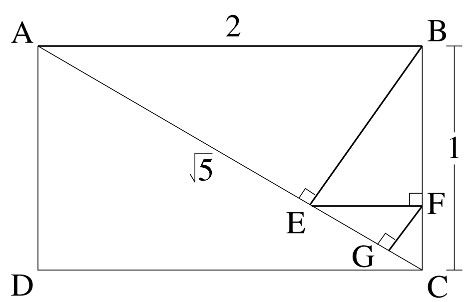

Figure 5

(d) Show that the lengths of line segments $\overline{AB}, \overline{BE}, \overline{EF}$, and $\overline{FG}$ form a geometric sequence by finding the common ratio. If we added on the lengths of the rest of the line segments leading to point C, we would have an infinite geometric series. Find the sum of that series.

(e) Add the length of diagonal $\overline{AC}$ to the sum of your geometric sequence to find the total length of point Les's trip. What is it? Take a moment to consider how amazing it is that an infinite number of segments has a total length which is finite!

3. Compare the lengths of Ed's and Les's trips. Which one is longer?

4. Let's look at a general rectangle. Let rectangle ABCD have a diagonal of length one.[48] Let $\angle BAC$ have measure θ. (See Figure 6.)

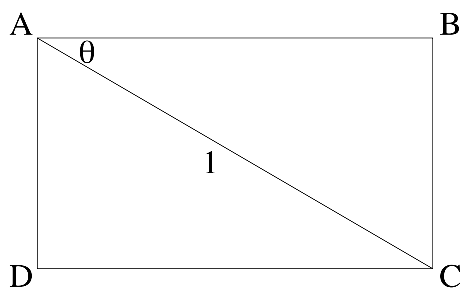

Figure 6

(a) First, consider point Ed's trip. In terms of θ, what is the length of $\overline{AB}$? What is the length of $\overline{BC}$? Give the distance around the sides of this rectangle as a function of θ.

(b) Next, consider point Les's trip.

i. Find the lengths of $\overline{AB}, \overline{BE}, \overline{EF}$, and $\overline{FG}$ for the general rectangle. (Refer back to Figure 2.)

ii. If we added on the lengths of the rest of line segments leading to point C, we would have an infinite geometric series. Find the sum of this series as a function of θ.[49] Add the length of the diagonal, $\overline{AC}$, to your function.

[48] This is still a general rectangle, since what is important is the *ratio* of the two sides instead of the actual lengths. Therefore, we need a general $\angle BAC$, but can specify the length of the diagonal. We chose $AC = 1$ to make our task easier.

[49] Finding a formula for an infinite sum is referred to as "writing the sum in closed form" by mathematicians.

5. In the example at the beginning of this project, you should have found that point Ed's trip was longer. We are interested in determining whether or not this is always true, using the general rectangle.

(a) Graph both the function giving the total length of Ed's trip and the function giving the total length of Les's trip on the same set of axes for $0^\circ < \theta < 90^\circ$.

(b) When is Ed's trip shorter than Les's trip? When is Ed's trip longer?

(c) Find the angle at which the two trips are the same. Your graph gives you a good way to approximate this angle. Using your symbolic formula, demonstrate that you have the exact value where this occurs.